§10 Two-dim Chiral QFT - II

__Last time__: $\varphi \in \Omega^{0,*}(\Sigma, h)$ $\beta\gamma$-bc system

$$\frac{1}{2}\int \langle \varphi, \bar\partial \varphi \rangle + \underbrace{\int \mathcal{L}(\partial_z^{\bullet} \varphi)}_{= I}$$

I chiral interaction

Then ① the theory is UV finite

② Effectively renormalized QME

$$\Longleftrightarrow \quad [\oint \mathcal{L}, \oint \mathcal{L}] = 0.$$

- Regularized Integral and UV finiteness

The propagator "$= \bar\partial^{-1}$" $\sim$ Szegö Kernel which exhibits hol. pole $\frac{1}{z-w}$ along the diagonal

In general, the Feynman Diagram involves

$$\int_{\Sigma^n} \Omega$$

where Ω exhibits hol. poles of arbitrary order when $z_i \to z_j$.

It turns out that such looking divergent integral has an intrinsic regularization via its conformal structure.

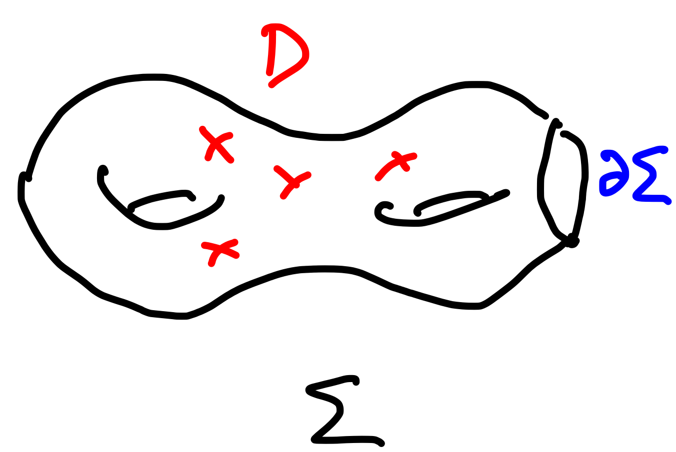
For simplicity, we start by considering such an $\int$

$$\int_\Sigma \omega$$

Here Σ is a Riemann Surface, possibly w/ boundary $\partial\Sigma$.

ω is a 2-form on Σ w/ meromorphic poles of arbitrary orders along a finite set $D \subset \Sigma$, $D \cap \partial\Sigma = \phi$.

Let $p \in D$, and z be a local coordinate centered at p.



Then locally ω can be written as

$$\omega = \frac{\eta}{z^n}$$ where η is smooth and $n \in \mathbb{Z}$.

Since the pole order can be arbitrarily large, the naive $\int_\Sigma \omega$ is divergent in general.

One intrinsic way-out (L-Zhou 2020)

We can decompose ω into

$$\omega = \alpha + \partial\beta \quad \text{where}$$

- α is a 2-form with at most logarithmic pole along D
- β is a $(0,1)$-form with arbitrary order of poles along D
- $\partial = dz\frac{\partial}{\partial z}$ is the holomorphic de Rham.

Rk: Such decomposition exists and NOT unique.

Def'n [L-Zhou] Define the regularized integral by

$$⨍_\Sigma \omega := \int_\Sigma \alpha + \int_{\partial\Sigma}\beta$$

Ref: [L-Zhou] Regularized integrals on Riemann surfaces and modular forms. (CMP 2021)

- It does NOT depend on the choice of α, β.
- $\fint_\Sigma$ is invariant under conformal transformations.
- $\fint_\Sigma \partial(-) = \int_{\partial\Sigma} (-)$
- $\fint_\Sigma \bar\partial(-) = \mathrm{Res}(-)$

$$\begin{tikzcd}
\mathcal{A}^2(\Sigma) \arrow[rr, hook] \arrow[dr, "\int_\Sigma"'] & & \mathcal{A}^2(\Sigma, *D) \arrow[dl, "\fint_\Sigma"] \\
& \mathbb{C} &
\end{tikzcd}$$

We can use this to define integrals on configuration spaces

$$\mathrm{Conf}_n(\Sigma) = \Sigma^n - \Delta$$
$$= \{ (p_1, \cdots, p_n) \in \Sigma^n \mid p_i \neq p_j, \forall i \neq j \}$$

and define

$$\fint_{\Sigma^n} : \mathcal{A}^{2n}(\Sigma^n, *\Delta) \longmapsto \mathbb{C} \quad \text{by}$$

$$\int_{\Sigma^n}(-) = \int_{\Sigma_1}\int_{\Sigma_2}\cdots\int_{\Sigma_n}(-)$$

- It does NOT depend on the choice of the ordering of the factors in Σ^n: Fubini-type theorem.

This gives an intrinsically regularized meaning for

$$\int_{\Sigma^n}\Omega \rightsquigarrow \text{Feynman Diagram integrand.}$$

This explains why the theory is UV-finite.

- Homological structure of BV quantization

Roughly speaking, BV quantization in QFT leads to

- Factorization algebra Obs of observables (Costello-Gwilliam)
- $(C_\cdot(\mathrm{Obs}), d)$: a chain complex via algebraic structures of Obs.
- A BV algebra $(\mathcal{A}, \Delta)$ describing the zero modes

(this is at $L=\infty$) w/. a BV-$\int$ map

$$\int_{BV} : \mathcal{A} \longmapsto \mathbb{C}$$

- A $\mathbb{C}[[\hbar]]$-linear map

$$\langle - \rangle : C_\bullet(\text{obs}) \longmapsto \mathcal{A}((\hbar))$$

Satisfies the following QME

$$(d+\hbar\Delta)\langle - \rangle = 0$$

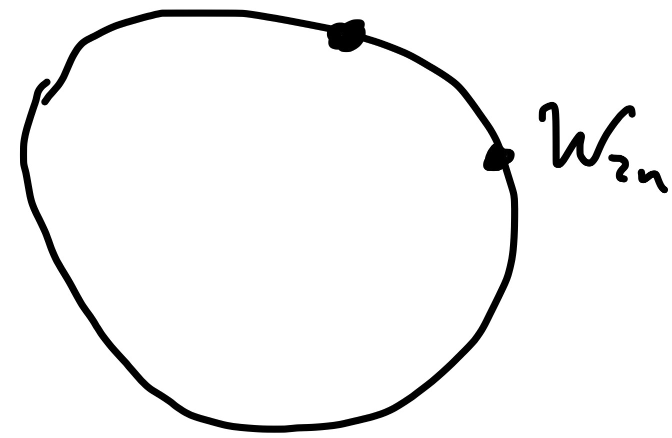
($\langle - \rangle$ is the HRG from $L=0$ to $L=\infty$.)

QME says that $\langle - \rangle$ is a chain map intertwining d and $\hbar\Delta$.

- Partition function.

$$\text{Index} = \int_{BV} \langle 1 \rangle$$

In the example of TQM



- $\mathrm{Obs} = \mathcal{W}_{2n}$ Weyl algebra
- $(C_\bullet(\mathrm{Obs}), d)$ = Hochschild chain complex
- BV algebra on zero modes $(\mathcal{A}, \Delta) = (\Omega^\bullet(\mathbb{R}^{2n}), \mathcal{L}_{\omega^{-1}})$
- $\langle - \rangle$ = free correlation map

$$\langle - \rangle : \underset{b}{C_\bullet(\mathcal{W}_{2n})} \longmapsto \underset{\mathcal{L}_{\omega^{-1}}}{\Omega^\bullet(\mathbb{R}^{2n})((\hbar))}$$

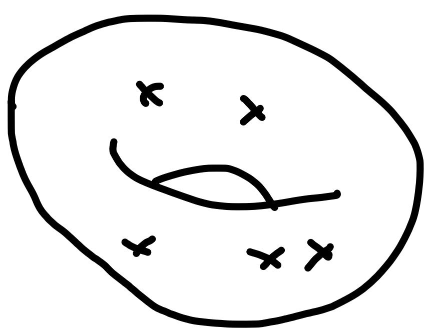
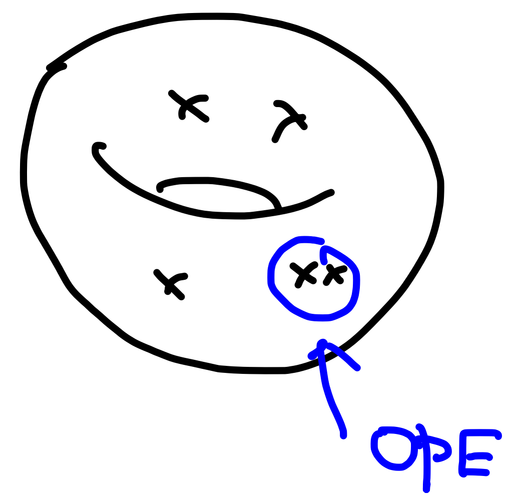
- $\mathrm{Index} = \int_{BV} \langle 1 \rangle = \left[e^{\omega_\hbar/\hbar} \hat{A} \right]$

In the example of 2d Chiral, we have a similar story

Ref: [Gui-L]: Elliptic trace map on Chiral algebras. arXiv: 2112.14572 [math.QA]

- Beilinson–Drinfeld's Chiral Chain Complex

Intuitively, Chiral chain complex can be viewed as a 2d Chiral analogue of Hochschild Chain Complex

d_{ch}  $= \sum$ 

- [Zhu 1994] studies the space of genus 1 conformal blocks (the 0-th elliptic chiral homology)
- [Beilinson–Drinfeld] Chiral homology for general algebraic curves.
- [Ekeren–Heluani 2018, 2021] Explicit complex expressing the 0th and 1th elliptic Chiral homology

The construction of Beilinson - Drinfeld:

- $\mathcal{M}(X)$: category of right D-module on $X = \Sigma$
- $\mathcal{M}(X^S)$: category of right D-module on X^S

$M \in \mathcal{M}(X^S) \Longleftrightarrow$ for each finite index set $I \in S$,

assign a right D-module M_{X^I} on X^I

(satisfying some compatibility conditions)

- There is an exact fully faithful embedding

$$\Delta_*^{(S)} : \mathcal{M}(X) \hookrightarrow \mathcal{M}(X^S)$$

via the diagonal map $\Delta^{(I)} : X \hookrightarrow X^I$.

- $\mathcal{M}(X^S)$ carries a tensor structure $\otimes^{ch}$.

Then a chiral algebra $\mathcal{A}$ is a

Lie algebra object via $\Delta_*^{(S)}$.

<u>RK</u>. This collects all "normal ordered operators".

We consider the Chevalley-Eilenberg complex

$$\left(C(\mathcal{A}), d_{CE}\right) = \left(\bigoplus \mathrm{Sym}^{\bullet}_{\otimes^{ch}}(\Delta_*^{(S)} \mathcal{A}[1]), d_{CE}\right)$$

The chiral homology (complex)

$$C^{ch}(X, \mathcal{A}) = R\Gamma_{DR}(X^S, C(\mathcal{A}))$$

We will focus on $\beta\gamma$-bc system, the VOA

$\mathcal{V}^{\beta\gamma\text{-}bc} \rightsquigarrow$ chiral algebra $\mathcal{A}^{\beta\gamma\text{-}bc}$

Thm [Gui-L] Let E be an elliptic curve. Then the homotopic RG flows gives a map

$$\langle - \rangle_{2d} : C^{ch}(E, \mathcal{A}^{\beta\gamma\text{-}bc}) \longmapsto \mathbb{A}((\hbar))$$

satisfying QME

$$(d_{ch} + \hbar\Delta)\langle - \rangle_{2d} = 0$$

Roughly speaking, $\langle - \rangle$ is defined by

$$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} := \fint_{E^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$$

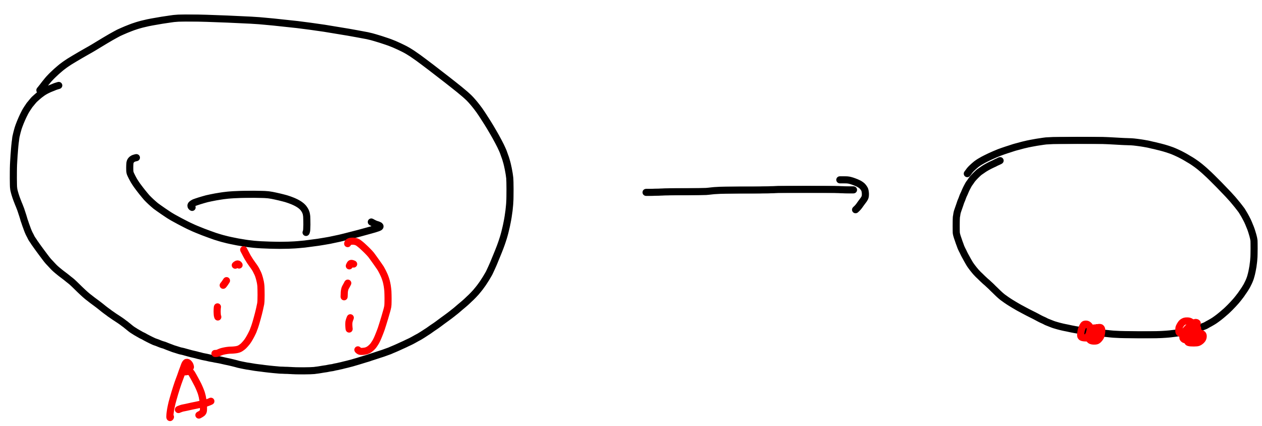
where

- $\langle \mathcal{O}(z_1) \cdots \mathcal{O}(z_n) \rangle$ is local correlator given by Feynman diagrams
- $\fint_{E^n}$ is the regularized integral.

1d TQM	2d chiral QFT
Associative algebra	Vertex algebra/Chiral alg.
Hochschild homology	Chiral homology
QME $(\hbar\Delta + b)\langle - \rangle_{1d} = 0$	QME $(\hbar\Delta + d_{ch})\langle - \rangle_{2d} = 0$
$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{1d} = \int_{\overline{\mathrm{Conf}_n(S^1)}}$	$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} = \fint_{\Sigma^n}$

• $2d \to 1d$ Reduction

In physics, partition function / correlation function on elliptic curves are described by QM on S^1



$$\langle \quad \rangle_{2d} \longrightarrow \mathrm{Tr}_{\mathcal{H}} \langle \cdots \rangle$$

Now we can define 2d correlation functions using regularized integral $\fint_E$. In 1d, operators are described by A-cycle $\oint_A$. These two are not exactly the same, but related via each other by holomorphic anomaly.

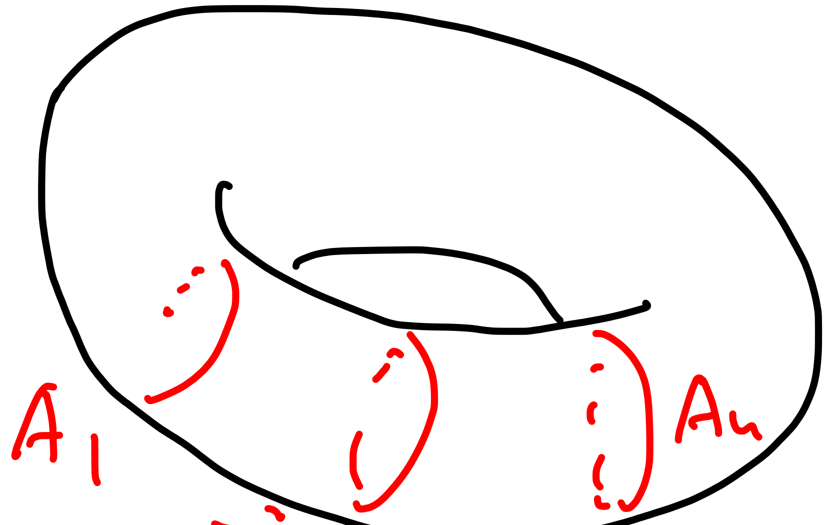
Thm [L-Zhou] Let $\Phi(z_1,\cdots,z_n;\tau)$ be a meromorphic elliptic function on $\mathbb{C}^n\times\mathbb{H}$ which is holomorphic away from diagonals. Let $A_1,\cdots,A_n$ be n disjoint A-cycles on $E_\tau=\mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}$.

Then the regularized integral

$$\fint_{E_\tau^n}\left(\prod_{i=1}^n\frac{d^2z_i}{\operatorname{Im}\tau}\right)\Phi(z_1,\cdots,z_n;\tau)$$

lies in $\mathcal{O}_{\mathbb{H}}\left[\frac{1}{\operatorname{Im}\tau}\right]$. Moreover, we have

$$\lim_{\tau\to\infty}\fint_{E_\tau^n}\left(\prod_{i=1}^n\frac{d^2z_i}{\operatorname{Im}\tau}\right)\Phi=\frac{1}{n!}\sum_{\sigma\in S_n}\int_{A_{\sigma(1)}}dz_1\cdots\int_{A_{\sigma(n)}}dz_n\,\Phi$$



$$\fint_{E^n} \xrightarrow{\lim \bar{\tau} \to \infty} \text{averaged } \int_A$$

almost holomorphic modular quasi-modular.

The anti-holomorphic dependence has a precise description

<u>Thm</u> [L-Zhou] Let Φ be an almost-elliptic function. Then one has

$$\partial_Y \fint_{E_\tau^n} \Phi = \fint_{E_\tau^n} \partial_Y \Phi - \sum_{i<j} \fint_{E_\tau^{n-1}} \operatorname{Res}_{z_i = z_j} \left((z_i - z_j) \Phi \right)$$

This gives the holomorphic anomaly equation.